University of Saskatchewan
Department of Mathematics and Statistics
Math 338

March 31, 2006 | Midterm Examination #3 | 50 minutes

Fully answer all the questions in the booklets provided. Permitted resources: the attached table of Laplace transforms.

This is a midterm examination. Cheating on an examination is considered a serious offense by the University and can be met with disciplinary action, including suspension or expulsion. Candidates shall not bring into the examination room any books, resources or papers except at the discretion of the examiner or as indicated on the examination paper. Candidates shall hold no communication of any kind with other candidates within the examination room.

Question 1. Using the basic definition of the Laplace transform, which is **5**

$$\mathcal{L}(f) = \int_0^\infty f(t)e^{-st}\,dt,$$

show that

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

Question 2. Sketch the odd extension of the function **5**

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \le x < \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$$

and determine the Fourier sine series of f up to and including the first four terms.

Question 3. An object of mass $m > 0$ is attached to a spring which has spring-constant $k > 0$. **5**
Initially the object is at rest but at time $t = 1$ it is struck, so that the initial value problem for its motion is

$$\frac{d^2y}{dt^2} + \omega_0^2 y = A\,\delta(t-1), \quad y(0) = 0, y'(0) = 0,$$

where A is some constant and $\omega_0^2 = k/m$. Calculate: the velocity of the object at time $t = 2$ divided by the velocity of the object immediately after it is struck.

Question 4. Find a function $f(t)$ whose Laplace transform is **5**

$$F(s) = \frac{1}{s^4 - 3s^2 - 4}.$$

TEST 3 SOLUTIONS

1 $$\mathcal{L}(f'') = \int_0^\infty f''(t)e^{-st}\,dt$$

$$= f'(t)e^{-st}\Big|_0^\infty - \int_0^\infty f'(t)(-s)e^{-st}\,dt$$

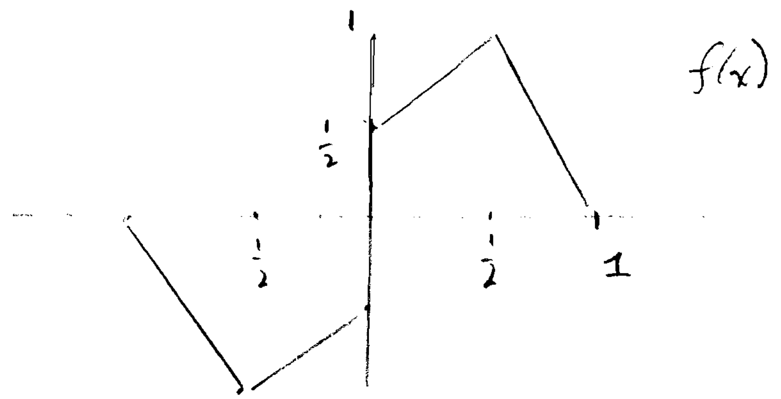
$$= -f'(0) + s\int_0^\infty f'(t)e^{-st}\,dt$$

$$= -f'(0) + s\left(f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)(-s)e^{-st}\,dt \right)$$

$$= -f'(0) - s\,f(0) + s^2\int_0^\infty f(t)e^{-st}\,dt$$

$$= \mathcal{L}(f) - s\,f(0) - f'(0)$$

2



$L = 1$, $b_n = \frac{2}{L}\int_0^1 f(x)\sin n\pi x\,dx$

$$= 2\left(\int_0^{\frac{1}{2}} \left(x+\frac{1}{2}\right)\sin n\pi x + \int_{\frac{1}{2}}^{1} 2(1-x)\sin n\pi x\right)$$

$$= 2\left(\int_0^{\frac{1}{2}} \left(x+\frac{1}{2}\right) d\left(\frac{\cos n\pi x}{-n\pi}\right) + \int_{\frac{1}{2}}^{1} 2(1-x)\, d\left(\frac{\cos n\pi x}{-n\pi}\right)\right)$$

$$= 2\left(\left(x+\frac{1}{2}\right)\frac{\cos n\pi x}{-n\pi}\Bigg|_0^{\frac{1}{2}} + \frac{1}{n\pi}\int_0^{\frac{1}{2}} \cos n\pi x\right.$$

$$\left. + 2(1-x)\frac{\cos n\pi x}{-n\pi}\Bigg|_{\frac{1}{2}}^{1} - \frac{1}{n\pi}\int_{\frac{1}{2}}^{1} 2\cos n\pi x\right)$$

$$= 2\left(-\frac{1}{n\pi}\cos\frac{n\pi}{2} + \frac{1}{2n\pi} + \frac{\cos\frac{n\pi}{2}}{n\pi}\right.$$

$$\left. + \frac{1}{n^2\pi^2}\sin n\pi x\Bigg|_0^{\frac{1}{2}} - \frac{2}{n^2\pi^2}\sin n\pi x\Bigg|_{\frac{1}{2}}^{1}\right)$$

$$= \frac{1}{n\pi}$$

$$+ \frac{2}{n^2\pi^2}\left(\sin\frac{n\pi}{2} + 2\sin\frac{n\pi}{2}\right)$$

$$= \frac{1}{n\pi} + \frac{6}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{1}{\pi}\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \right) + \frac{6}{\pi^2}\left(1, 0, \frac{-1}{9}, 0, \cdots\right)$$

$$= \left(\frac{1}{\pi} + \frac{6}{\pi^2}, \frac{1}{2\pi}, \frac{1}{3\pi} - \frac{2}{3\pi^2}, \frac{-1}{4\pi}, \cdots\right)$$

The FS is

$$f = \left(\frac{1}{\pi} + \frac{6}{\pi^2}\right)\sin \pi x + \frac{1}{2\pi}\sin 2\pi x + \left(\frac{1}{3\pi} - \frac{2}{3\pi^2}\right)\sin 3\pi x - \frac{1}{4\pi}\sin 4\pi x + \cdots$$

3 /

$$s^2 Y + w_0^2 Y = A \int_0^\infty e^{-st} \delta(t-1) = A e^{-s}$$

$$Y = \frac{A e^{-s}}{s^2 + w_0^2}$$

$$y(t) = A \quad u(t-1) \quad \mathcal{L}^{-1}\left(\frac{1}{s^2 + w_0^2} \right)\Big|_{t-1}$$

$$= A \; u(t-1) \; \frac{1}{w_0} \; \sin\left(w_0 (t-1) \right)$$

For $t > 1$, $y' = \frac{d}{dt}\left(\frac{A}{w_0} \sin\left(w_0 (t-1) \right) \right) = A \cos w_0 (t-1)$, so

the required number is $\frac{A \cos w_0 (2-1)}{A \cos w_0 (1-1)} = \cos w_0$

4 /

$$\frac{1}{s^4 - 3s^2 - 4} = \frac{1}{(s^2-4)(s^2+1)} = \frac{1}{(s-2)(s+2)(s^2+1)} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{Cs+D}{s^2+1}$$

$$1 = A(s+2)(s^2+1) + B(s-2)(s^2+1) + (Cs+D)(s-2)(s+2)$$

$s = 2$ $\quad 1 = A(4)(5)$ $\quad A = \frac{1}{20}$

$s = -2$ $\quad 1 = B(-4)(5)$ $\quad B = \frac{-1}{20}$

$s=0$: $1 = A\cdot(2)(1) + B(-2)(1) + D(-2)(2)$

$= \frac{1}{5} - 4D \qquad D = -\frac{1}{5}$

$s=1$: $1 = A(3)(2) + B(-1)(2) + \left(C - \frac{1}{5}\right)(-1)(3)$

$= \frac{6}{20} + \frac{2}{20} - 3C + \frac{3}{5} = 1 - 3C \qquad C = 0$

$$\frac{1}{s^4 - 3s^2 - 4} = \frac{1}{20(s-2)} - \frac{1}{20(s+2)} - \frac{1}{5(s^2+1)}$$

$$f(t) = \frac{1}{20} e^{2t} - \frac{1}{20} e^{-2t} - \frac{1}{5} \sin t$$

$$= \frac{1}{10} \sinh 2t - \frac{1}{5} \sin t$$